2 Photogrammetric networks

2.1 The concepts of precision and reliability

2.1.3 Notations, definitions

Let the linear statistical model of bundle adjustment be:

$$\ell - e = Ax;\ P$$
$$E(\ell) = Ax;\quad E(e) = 0$$
$$D(e) = E(ee^T) = \sigma_0^2 P^{-1}, \quad D(\ell) = \sigma_0^2 P^{-1}$$

e … vector of true observation errors
P ... weight matrix of the observation vector ℓ
σ_0^2.... standard deviation of unit weight to be estimated

For the purpose of interval estimation we assume

$$\overline{H}_0: \quad \ell \sim N\ (Ax,\ \sigma_0^2\ P^{-1})$$

(i.e. ℓ has a multidimensional normal distribution with the expectation $E(\ell) = Ax$ and dispersion $D(\ell) = \sigma_0^2\ P^{-1}$).

Suppose a minimum variance unbiased estimation of x and σ_0^2 with

$$\hat{x} = (A^T PA)^{-1} A^T P\ell$$
$$\hat{\sigma}_0^2 = \frac{1}{r}(A\hat{x} - \ell)^T\ P(A\hat{x} - \ell) \qquad ; \qquad r = n - u\ \ \text{(redundancy)}$$

and denote the residuals with

$$v = A\hat{x} - \ell = \left(A\left(A^T PA\right)^{-1} A^T P - I\right)\ell\ ,$$

then under $\overline{H}_0$ the distributions of $\hat{x}$ and v are

$$\hat{x} \sim N\left(x,\ \sigma_0^2\, Q_{xx}\right), \quad Q_{xx} = \left(A^T PA\right)^{-1}$$
$$v \sim N\left(0,\ \sigma_0^2\, Q_{vv}\right), \quad Q_{vv} = P^{-1} - AQ_{xx}A^T$$

Generally the stochastic properties of adjustment results, which are collected in a vector h

$$h = \begin{bmatrix} \hat{x} \\ \hat{\ell} \\ v \\ f \end{bmatrix} = \begin{bmatrix} N^{-1}A^TP \\ AN^{-1}A^TP \\ AN^{-1}A^TP - I \\ FN^{-1}A^TP \end{bmatrix} \ell = H\ell$$

are given by

$$\Sigma_{hh} = \sigma_0^2 Q_{hh} = \sigma_0^2 H Q_{\ell\ell} H^T$$

$$Q_{hh} = \begin{bmatrix} Q_{xx} & Q_{x\hat{\ell}} & Q_{xv} & Q_{xf} \\ Q_{\hat{\ell}x} & Q_{\hat{\ell}\hat{\ell}} & Q_{\hat{\ell}v} & Q_{\hat{\ell}f} \\ Q_{vx} & Q_{v\hat{\ell}} & Q_{vv} & Q_{vf} \\ Q_{fx} & Q_{f\hat{\ell}} & Qv & Q_{ff} \end{bmatrix} = \begin{bmatrix} N^{-1} & N^{-1}A^T & 0 & N^{-1}F^T \\ AN^{-1} & AN^{-1}A^T & 0 & AN^{-1}F^T \\ 0 & 0 & P^{-1} - AN^{-1}A^T & 0 \\ FN^{-1} & FN^{-1}A^T & 0 & FN^{-1}F^T \end{bmatrix}$$

The accuracy (quality of an adjustment system, thus of the estimated quantities) consists of two parts: PRECISION AND RELIABILITY.

PRECISION: Describes the stochastical properties of estimated quantities if the a-priori assumptions (functional and stochastical relations) are considered to be true.

RELIABILITY: Describes the quality and the sensitivity of the adjustment model with respect to the detection of model errors (blunders, systematic errors, stochastical errors). For further details refer to Gruen, 1982, pp. 46-47, PERS 1/82.

2.1.2 Precision criteria for x, y-networks

Local precision criteria

With $\sigma_0^2 = 1$: $Q_{xx} = \Sigma_{xx}$

Points i, j:

$$Q_{xx(ij)} = \begin{bmatrix} Q_{ii} & Q_{ij} \\ Q_{ji} & Q_{jj} \end{bmatrix}$$

Q_{ii} contains information on precision of point P_i

Q_{ij} contains information on precision between points P_i , P_j

Relative precision: $Q_{\Delta ij} = Q_{ii} + Q_{jj} - Q_{ij} - Q_{ji}$

Shows precision structure of coordinate differences

a) Point related criteria

Spectral decomposition of Q_{ii}:

$$Q_{ii} = V_{ii} \Lambda_{ii} V_{ii}^T \quad , \quad \text{with } \Lambda_{ii} = \text{diag}\,(\lambda_1, \lambda_2)_{ii} \;\; , \;\; \lambda_1 \geq \lambda_2$$

V_{ii} contains eigen vectors (orthonormalized)

With axes $a_{ii} = \left(2\sqrt{\lambda_1}\right)_{ii}$, $b_{ii} = \left(2\sqrt{\lambda_2}\right)_{ii}$ we obtain Helmert's error ellipse

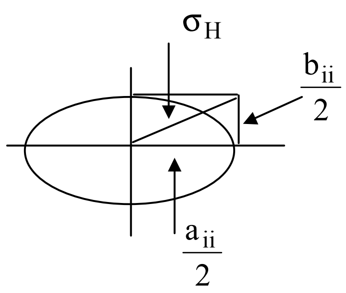


Fig. 2.1: Helmert's error ellipse

Scalar measures:

Helmert's point error: $\sigma_H^2 = \text{tr}(Q_{ii}) = \left(\lambda_1 + \lambda_2\right)_{ii} = \frac{a_{ii}^2}{4} + \frac{b_{ii}^2}{4}$

Werkmeister's point error: $\sigma_w^2 = \det(Q_{ii}) = \left(\lambda_1 + \lambda_2\right)_{ii}$

Friedrich's point error:

$$\sigma_F^2 = \lambda_{max}(Q_{ii}) = (\lambda_1)_{ii}$$

Condition number:

$$\text{cond}(Q_{ii}) = \left(\frac{\lambda_1}{\lambda_2}\right)_{ii} \quad ; \quad \text{circle: } \text{cond}(Q_{ii}) = 1 \qquad \text{(isotropy)}$$

Modified point errors:

$$\sigma_{\overline{H}}^2 = \left(\frac{\lambda_1 + \lambda_2}{2}\right)_{ii} \quad , \quad \text{arithmetic mean}$$

$$\sigma_{\overline{W}}^2 = \left(\sqrt{\lambda_1 \cdot \lambda_2}\right)_{ii} \quad , \quad \text{geometric mean}$$

Principal disadvantage of scalar precision measures: direction of errors not visible !
Area related measures like ellipses cannot directly be compared with each other !

b) Relative precision measures

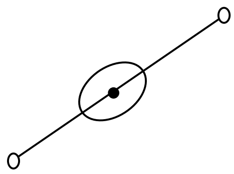
Relative error ellipse (Baarda):

$$Q_{\Delta ij} \rightarrow (\lambda_1, \lambda_2)_{\Delta ij} \quad , \quad (a,\ b)_{\Delta ij}$$

Scalar measures:

$$\text{tr}\left(Q_{\Delta ij}\right) = (\lambda_1 + \lambda_2)_{\Delta ij}$$
$$\det\left(Q_{\Delta ij}\right) = (\lambda_1 \cdot \lambda_2)_{\Delta ij}$$
$$\lambda_{Max}\left(Q_{\Delta ij}\right) = (\lambda_1)_{\Delta ij}$$

Bomford: mean distance, direction error



Graphical display of all corrections not possible, thus restriction to neighboring points.

Global precision criteria

Global precision criteria refer to the complete Q_{XX} - matrix.

They provide for average measures of the network as a whole.

Eigen values of the Q_{XX} - matrix (assumed to be positive semi definite):

$$Q_{xx} = V \Lambda V^T \quad \text{(spectral decomposition)}$$

$$\Lambda = \text{diag}\left(\lambda_1, \lambda_2, \ldots., \lambda_m\right)$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \quad ; \quad \lambda_{k+1} = \ldots = \lambda_m = 0$

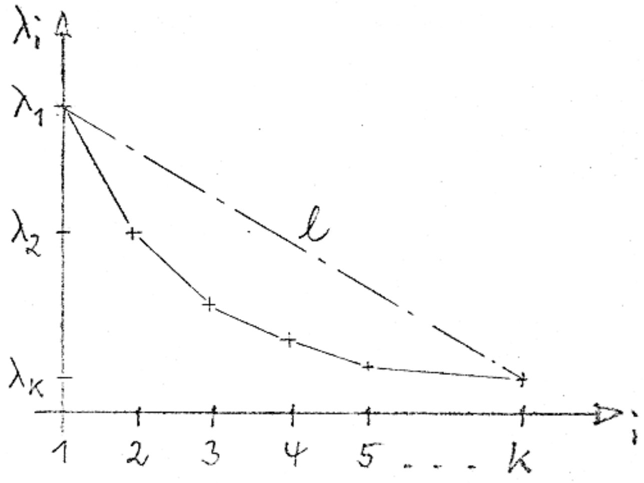
V contains the normalized eigen vectors:

$$VV^T = V^T V = I_{m,m}$$

The defect of Q_{XX} is $d = m - k$.

$2\sqrt{\lambda_i}$ $(i = 1, \ldots\ldots ,k)$ are the axes of a k-dimensional hyperellipsoid. The eigen vectors indicate the directions of these axes, they are orthogonal to each other.

If k>3 the geometrical representation of the hyperellipsoid is not possible any more.



ℓ... Linear declination of eigen values

Figure 2.2: Graphical representation of the eigen values

A generalization of the local precision measures gives the global measures.

For p network points (m = 2p) we obtain:

Generalized Helmert "point" error:

$$pM_H^2 = tr(Q_{xx}) = \lambda_1 + \lambda_2 + ... + \lambda_k$$

Generalized Werkmeister "point" error:

$$M_W^{2p} = \det(Q_{xx}^*) = \lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_k \;;$$

$$Q_{xx} \rightarrow \begin{bmatrix} Q_{xx}^* & 0 \\ 0 & 0 \end{bmatrix} \text{, with } Q_{xx}^* = UQ_{xx}U^T$$

Attention: det $(Q_{XX}) = 0$, if Q_{XX} singular !

$\sqrt{M_W^{2p}}$ is direct proportional to the volume of the k-dimensional hyperellipsoid.

Generalized Friedrich "point"error: $\lambda_{Max}\,(Q_{XX}) = \lambda_1$

Condition number: $\operatorname{cond}(Q_{XX}) = \dfrac{\lambda_{Max}}{\lambda_{Min}} = \dfrac{\lambda_1}{\lambda_k}$

Modified global measures:

$$M_H^2 = \frac{1}{k} tr\,(Q_{XX}) = \frac{1}{k}(\lambda_1 + \lambda_2 + ... + \lambda_k) \quad ; \text{ arithmetic mean}$$

$$M_W^2 = (\det\,(Q_{XX}))^{\frac{1}{k}} = (\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_k)^{\frac{1}{k}} \quad ; \text{ geometric mean}$$

Important measure: λ_{Max} (Attention: Dependent on dimension of Q_{XX} !)

$$\text{Cond}\left(Q_{xx}\right)=\frac{\lambda_{Max}}{\lambda_{Min}}$$ free of dimension problems !

Linear functions of x:

$f_i = F_i x$; F_i ... row vector of dimension 2p

With the Rayleigh – quotient we obtain

$$\lambda_{Min} \le \frac{F_i Q_{xx} F_i^T}{F_i F_i^T} \le \lambda_{Max} .$$

For the variances we get

$$\sigma_{f_i}^2 = \sigma_0^2 F_i Q_{xx} F_i^T .$$

Hence we obtain as limits for $\sigma_{f_i}^2$:

$$\sigma_0^2 F_i F_i^T \lambda_{Min} \le \sigma_{f_i}^2 \le \sigma_0^2 F_i F_i^T \lambda_{Max}$$

(Upper limit of $\sigma_{f_i}^2$ depends on λ_{Max} !)

The smaller the condition number the less differ the variances of estimable functions from each other. The network has a more homogeneous precision structure.

With normal equations $N = Q_{xx}^g$ we get

$$N = Q_{XX}^g = \left(V \Lambda V^T\right)^g = V \Lambda^g V^T \quad , \quad ^{1)}$$

with $\Lambda^g = \text{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k}, 0, \ldots, 0\right)$

Thus: $\text{cond}(N) = \dfrac{\frac{1}{\lambda_k}}{\frac{1}{\lambda_1}} = \dfrac{\lambda_1}{\lambda_k} = \text{cond}\left(Q_{xx}\right)$

[1] See Theorem 1.5.16 in **Graybill**: Theory and Application of the Linear Model, p.30

Global goal functions for network optimization:
(see Grafarend, Harland: Optimal design of geodetic networks I. In German. DGK, Reihe A, Heft 74, München 1973):

$$
\begin{aligned}
&\text{A - optimal:} && \operatorname{tr}(Q_{xx}) = \min\\
&\text{D - optimal:} && \det(Q_{xx}) = \min\\
&\text{E - optimal:} && \lambda_{\max}(Q_{xx}) = \min\\
&\text{I - optimal:} && \operatorname{cond}(Q_{xx}) = \min
\end{aligned}
$$

Transformation invariance of precision measures
(Congruence transformations: translation, rotation, reflection)

Translation :

$$\Sigma_{xx} = Q_{xx} = E\left[(x - E(x))\,(x - E(x))^T\right]$$

$$\bar{x} = x + \text{const.} \quad \succ \quad E(\bar{x}) = E(x) + \text{const}$$

$$
\begin{aligned}
\bar{\Sigma}_{xx} = \bar{Q}_{xx} &= E\left[(\bar{x} - E(\bar{x}))\,(\bar{x} - E(\bar{x}))^T\right]\\
&= E\left[(x - E(x))\,(x - E(x))^T\right] = Q_{xx}
\end{aligned}
$$

Rotation, reflection :

$$
\begin{aligned}
&\bar{x} = Tx \quad (T \text{ orthogonal})\\
&\bar{Q}_{xx} = TQ_{xx}T^T\\
&\operatorname{tr}(\bar{Q}_{xx}) = \operatorname{tr}(TQ_{xx}T^T) = \operatorname{tr}(Q_{xx}TT^T) = \operatorname{tr}(Q_{xx}) \ ; \ (\operatorname{tr}(AB) = \operatorname{tr}(BA))\\
&\det(\bar{Q}_{xx}) = \det(TQ_{xx}T^T) = \det(T)\det(T^T)\det(Q_{xx}) = \det(Q_{xx})
\end{aligned}
$$

Since all precision measures have been developed by using eigen values it is sufficient to prove the transformation invariance of the eigen values.

Translation: Proof trivial, since design matrix $A \ (N = A^TPA)$ contains only coordinate differences.

Rotation, reflection:

$$
\begin{aligned}
&\bar{x} = Tx \ ; \qquad \bar{Q}_{xx} = TQ_{xx}T^T\\
&\bar{Q}_{xx} = TV\Lambda V^TT^T = U\Lambda U^T, \ \text{with} \quad UU^T = U^TU = I\\
&\qquad\qquad\qquad\qquad TVV^TT^T = TVV^T = TIT^T = I
\end{aligned}
$$

Eigen vectors V get the same rotation (reflection) as the whole coordinate system.
For local precision measures we obtain an identical derivation $\succ$ eigen values (length of ellipse axes) are invariant; only rotation /reflection of eigen vectors ($\rightarrow$ direction of ellipse axes).

The same is valid for the relative error ellipses and for the boundary criterium:

$$f = Fx = \overline{F}\overline{x} = \overline{F}Tx; \quad \text{with} \quad \overline{F} = FT^{-1} \; ; \; \overline{F}^T = TF^T$$

$$\overline{Q}_{ff} = \overline{F}TQ_{xx}T^T\overline{F}^T = FT^{-1}TQ_{xx}T^TTF^T = FQ_{xx}F^T = Q_{ff}$$

2.1.3 Reliability criteria and the detection of blunders

2.1.3.1 General relations

Residuals $v = A\hat{x} - \ell$

With $\hat{x} = \left(A^TPA\right)^{-1} A^TP\,\ell$ we get

$$v = \left(A\left(A^TPA\right)^{-1} A^TP - I \right) \ell$$

or $v = -\,Q_{vv}P\,\ell$

With $\ell = Ax + e$;

$$v = -\left(Q_{vv}PAx + Q_{vv}Pe\right) = -Q_{vv}Pe$$

(since $Q_{vv}PA = 0$)

Assume gross errors $\nabla\ell$ in observations ℓ. $\nabla\ell$ is transformed to the residuals with

$$v + \nabla v = -Q_{vv}P\left(\ell + \nabla\ell\right)$$

and from this we get

$$\nabla v = -Q_{vv}P\,\nabla\ell$$

Important for reliability considerations:

$Q_{vv}P$ - matrix

With the adjusted observations $\hat{\ell}$

$$\hat{\ell} = \ell + v$$

We obtain

$$Q_{\hat{\ell}\hat{\ell}} = AN^{-1}A^T$$

and using $Q_{vv} = P^{-1} - AN^{-1}A^T$ we get

$$Q_{vv} = P^{-1} - Q_{\hat{\ell}\hat{\ell}}$$

Some characteristics of the Q_{VV} - matrix (if P=I)

The Q_{vv} matrix is *idempotent,* i.e.

$$Q_{vv} \cdot Q_{vv} = Q_{vv}$$

and it has the following characteristics:

- Q_{vv} is square, symmetric and singular
- The main diagonal elements denoted by q_{ii} have values between 0 and 1

  $$0 \le q_{ii} \le 1$$

- The sum of the main-diagonal elements is equal to the reduncancy r

  $$r = q_{11} + q_{22} + q_{33} + \ldots + q_{nn}$$

  i.e. $\quad \text{Trace}\,(Q_{vv}) = r$

- Each main-diagonal element q_{ii} is equal to the sum of squares of all elements in the same row (or column) including the diagonal element itself.

  $$q_{ii} = q_{i1}^2 + q_{i2}^2 + \ldots q_{ii}^2 + \ldots + q_{in}^2$$

- The off-diagonal elements

  $$q_{ij} = \sum_{k=1}^{n} q_{ik}\, q_{jk}$$

  $$0 \le |q_{ij}| \le \sqrt{q_{ii} q_{jj}}$$

Laan proved that the maximum absolute value of any off-diagonal elements is 0.5.

$$0 \le |q_{ij}| \le 0.5$$

Characteristics of $M = Q_{vv}P$, Q_{vv} :

For $P = I$: $\quad M = Q_{vv}$

$\succ$ Q_{vv} takes the same properties as M
(→ idempotancy, $\text{tr}(Q_{vv}) = \text{rank}(Q_{vv}) = r$)

From $M = Q_{vv}P = I - AN^{-1}A^TP$
it is easy to see that

$$M \cdot M = M$$

$\succ$ M is an idempotent matrix ; singular
Non-singular idempotent matrix: $I\ (I \cdot I = I)$

For regular P it follows from the idempotency of M:

$$\text{tr}\,(M) = \text{rank}\,(M) = r \quad ; \quad r = n - u \text{ (redundancy)}$$

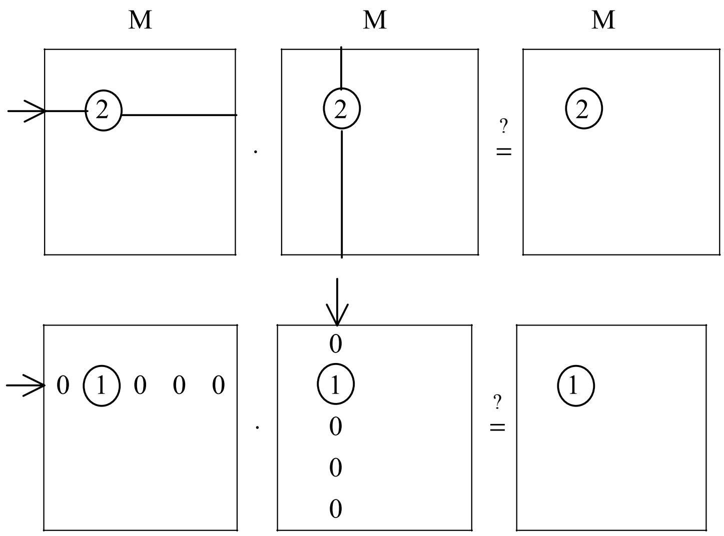
Q_{vv} is symmetric and singular (rank (Q_{vv}) = r) (for general P, Q_{vv} is not idempotent !)

Assume P = diag (p_1, p_2, ..., p_n):

M is symmetric

$0 \leq m_{ii} \leq 1$ (m_{ii} ... ith diagonal element of M)

Examples:



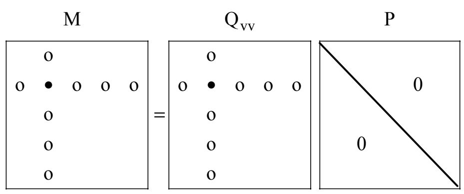
No !

Yes if all other row/column elements are = 0 !

If $m_{ii} = 1$: $m_{ij} = m_{ji} = 0$
($\text{for} \quad j = 1, \ldots, i-1, i+1, \ldots, n$)

If $m_{ii} = 0$: $m_{ij} = m_{ji} = 0$
($\text{for} \quad j = 1, \ldots, i-1, i+1, \ldots, n$)

If $m_{ii} = 0$: $q_{v_i v_i} = 0 \rightarrow q_{v_i v_j} = q_{v_j v_i} = 0$
($\text{for} \quad j = 1, \ldots, i-1, i+1, \ldots, n$



Since $Q_{vv} = MP^{-1}$

$$\begin{array}{ccccc} Q_{vv} & & M & & P^{-1} \\ \begin{bmatrix} & o & & & \\ o & o & o & o & o \\ & o & & & \\ & o & & & \\ & o & & & \end{bmatrix} & = & \begin{bmatrix} & o & & & \\ o & o & o & o & o \\ & o & & & \\ & o & & & \\ & o & & & \end{bmatrix} & & \begin{bmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix} \end{array}$$

$q_{v_i v_i} = 0$: "zero variance" situation

≻ gross error $\nabla \ell_i$ is not detectable !

$$\nabla v_i = -Q_{vv} P \nabla \ell_{(i)} = 0$$

$$\left(w_i = \frac{0}{0}, \text{ not defined} \right)$$

Example: Non-control point with 2 rays only

≻ Blunders in image coordinates which belong to the epipolar plane are undetectable (x-observations !)

Example: Photogrammetric intersection

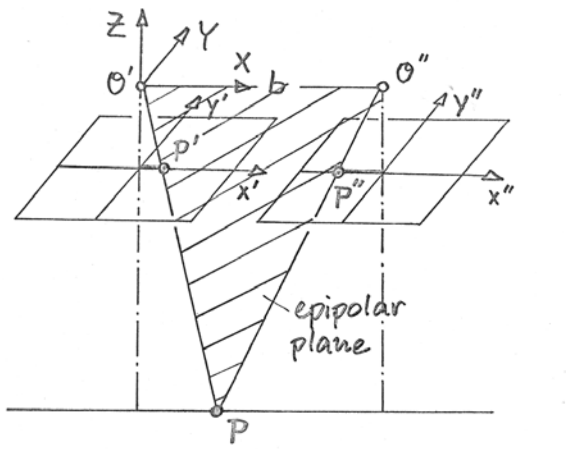


Fig. 2.3: Spatial intersection

Observations : $x', y', x'', y'' \rightarrow n = 4$

Unknowns: $X, Y, Z \rightarrow u = 3$

$\rightarrow \quad r = 1, \ \frac{r}{n} = \frac{1}{4}$

Normal case $(\varphi = \omega = \kappa = 0)$:

$$Z = \frac{-b \cdot c}{x' - x''}, \qquad y' = y'' = 0$$

$$Y = \frac{b \cdot y'}{x' - x''} \qquad \text{(without loss of generality)}$$

$$X = \frac{b \cdot x'}{x' - x''} \qquad p_x = x' - x'' \text{ (parallax)}$$

Functional model:

$$x' = -c\frac{X}{z}, \quad x'' = -c\frac{X-b}{z},$$

$$y' = -c\frac{Y}{Z}, \quad y'' = -c\frac{Y}{Z},$$

Coefficients of observation equations:

$$A = \begin{array}{c} \begin{matrix} dX & dY & dZ \end{matrix} \\ \begin{bmatrix} -\frac{c}{Z} & 0 & -\frac{x'}{Z} \\ 0 & -\frac{c}{Z} & 0 \\ -\frac{c}{Z} & 0 & -\frac{x''}{Z} \\ 0 & -\frac{c}{Z} & 0 \end{bmatrix} \end{array} \quad ; \quad P = I$$

$$N = \frac{1}{Z^2}\begin{bmatrix} 2c^2 & 0 & c\left(x' + x''\right) \\ & 2c^2 & 0 \\ & & x'^2 + x''^2 \end{bmatrix}; \quad N^{-1} = Z^2 \begin{bmatrix} \frac{x'^2 + x''^2}{c^2 \cdot px^2} & 0 & -\frac{x' + x''}{c \cdot px^2} \\ & \frac{1}{c^2} & 0 \\ & & \frac{2}{px^2} \end{bmatrix}$$

$$Q_{vv} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0.5 & 0 & -0.5 \\ & & 0 & 0 \\ & & & 0.5 \end{bmatrix}; \quad \mathrm{tr}\left(Q_{vv} I\right) = 1 = r$$

Correlations: $r_{y'y''} = -1.000....$

$$\nabla 1_{y'y''} = 5.8\ \sigma_0$$

$$\nabla 1_{x'x''0} \rightarrow \pm\infty$$

- Blunder detection: poor,
  Blunder location: impossible

2.1.3.2 Baarda's reliability theory

Baarda has developed a complete reliability theory, which is in practical use, e.g. in the Netherlands for geodetic network analysis for several years.
Currently the term "reliability" is mostly used in connection with gross error detection and location.
INTERNAL RELIABILITY: Defines the amount of a gross error in an observation, which is just non-detectable on a certain probability level.
EXTERNAL RELIABILITY: Indicates the effect of this non-detectable blunder on the estimated quantities (e.g. object point coordinates).

Data Snooping

Since true errors e of an adjustment system are not available, a gross error test procedure must be restricted on $\hat{x}$ or v.

Hence the expectations

$$E(\hat{x}) = x$$
$$E(v) = 0$$

have to be tested.

The expectation x for the solution vector (or at least a part of it) is usually not known (possible exceptions: control and check point coordinates).

Hence most of the observations are only to test by the null-hypothesis:

$$H_0 \; : \; E(v) = 0$$

However, this global hypothesis is not suitable to detect gross errors in individual observations.

If we suppose a sequence of null-hypotheses:

$$H_{0_i} : E(v_i) = 0, \qquad i = 1, \ldots, n$$

and a weight matrix P of diagonal form, then, according to a proposal of *Baarda* the statistics:

$$w_i = \frac{-v_i}{\sigma_{v_i}}, \text{ with } \sigma_{v_i}^2 = \sigma_0^2 q_{v_i v_i} \quad , \quad q_{v_i v_i} \text{ ... ith diagonal element of } Q_{vv}$$

should be used for testing.

If H_{0_i} us true, then w_i is distributed as Student's t:

$$w_i \sim t\,(1\text{-}\alpha_0\,,\infty) = n \text{ (normal)}, \qquad n(0,\,1)$$
$$\alpha_0 \text{... type I error size}$$
$$\alpha_0 = P\left(\left|w_i\right| > t(1-\alpha_0, \infty) / H_{0_i}\right)$$

This sequence of tests is called the "data-snooping" technique.

Since σ_{v_i} is usually not available, one has to use in practice very often :

$$w_i = \frac{-v_i}{\hat{\sigma}_{v_i}}, \qquad \text{with } \hat{\sigma}^2_{v_i} = \hat{\sigma}^2_0 q_{v_i v_i}$$

and $\quad w_i \sim t\,(r)$ under H_{0_i} (if v_i is independent of $\hat{\sigma}_0$)

If v_i is estimated from the same model as $\hat{\sigma}^2_0$:

$$\tau_r = \frac{v_i}{\hat{\sigma}_{v_i}} \sim \tau \quad (\text{under } H_{0_i});$$

Pope, 1975: The statistics of residuals and detection of outliers. XVIth General Assembly of
the IAG, Grenoble.

The influence of a gross error vector $\nabla\ell$ on the vector of residuals is

$$\nabla v = -Q_{vv} P \nabla\ell$$

If we suppose only one gross error $\nabla\ell_i$ in the ith observation we obtain the test criterion:

$$w_i = -\frac{v_i + \nabla v_i}{\sigma_{v_i}} \sim \quad n(\delta_i\,,1)$$

which enables the formulation of the alternative hypothesis

$$H_{A_i}: \; E(w_i) = \delta_i = -\frac{\nabla v_i}{\sigma_{v_i}} = \frac{q^{(i)}_{vv} \cdot P \cdot \nabla\ell_{(i)}}{\sigma_{v_i}} \qquad \nabla\ell_i = \begin{bmatrix} 0 \\ \vdots \\ \nabla\ell_i \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} 1 \\ \vdots \\ i \\ \vdots \\ n \end{matrix}$$

$q^{(i)}_{vv}$ ... ith row of Q_{vv}

with diagonal P the non-centrality parameter δ_i results in:

$$\delta_i = \frac{(q_{v_i v_i})^{1/2} p_i \nabla\ell_i}{\sigma_0} \quad . \tag{1}$$

This indicates the valuable relationship between the gross error $\nabla\ell_i$ and the non-centrality parameter δ_i.
Figure 2.4 shows the graphical representation of the test.

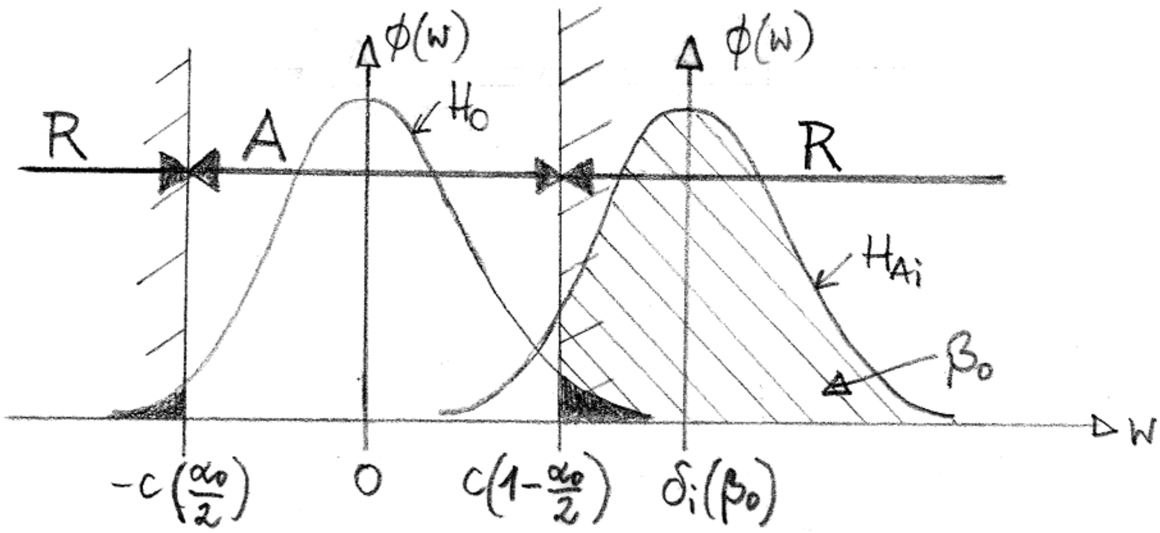


Fig. 2.4: Graphical representation of data snooping

H_0 …….. null-hypothesis
H_{A_i} …… alternative hypothesis
$\phi(w)$ …… density function
δ_i ……… non-centrality parameter
α_0 …….. type I error size
$1-\beta_0$ ….. type II error size (β_0 … power of the test)

A …….. acceptance region
R …….. rejection region
c ……... critical value for rejection (from Student's t)

Usually the alternative hypothesis is set up by using a specified non-centrality parameter and the corresponding power of the test β_0 is computed. Baarda goes the opposite way. He introduces a special β_0 (e.g. 0.8) for all alternative hypotheses H_{A_i} and computes the corresponding δ_i, which now can be considered as δ (common for the tests of all observations).
Nomograms for these values are to be found in Baarda (1968).

Example:

$\alpha_0 = 0.001 \qquad \rightarrow c = 3.3$

$\beta_0 = 0.8 \qquad \rightarrow \delta = 4.14$

Usually photogrammetric aerial systems have a large redundancy so that the proceeding considerations and Baarda's nomograms, which refer to a degree of freedom ∞ (for σ_0), can also be used if only $\hat{\sigma}_0$ is known.

Internal reliability

Global criterium: $\frac{r}{n}$ (relative redundancy)

With $r = \text{tr}\left(Q_{vv}P\right) = \text{rank}\left(Q_{vv}P\right)$

Local criteria: $r_i = \left(Q_{vv}P\right)_{ii}$; "redundancy number"

"local redundancy"

Assumption of one gross error $\nabla\ell_i$:

$$\nabla v = -r\nabla\ell_{(i)}$$

$$\begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \end{array}
\begin{array}{c} \nabla v \\ \begin{bmatrix} \\ X \\ \\ \\ \\ \\ \end{bmatrix} \end{array}
= -
\begin{array}{c} Q_{vv}P \\ \begin{bmatrix} & & & & & \\ & r_i & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}
\begin{array}{c} \nabla\ell_{(i)} \\ \begin{bmatrix} 0 \\ X \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}
\begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \end{array}$$

$$\nabla v_i = -r_i\nabla\ell_i$$

➢ The lager r_i the better the local internal reliability !

Criterium for good internal reliability:

$$\inf\{r_i\} = \max \qquad (i = 1, \ldots\ldots, n)$$

Given r, this is the case if all r_i ($0 \le r_i \le 1$) do have equal size:

$$r_m = \frac{r}{n} = \frac{1}{n}\sum_{i=1}^{n} r_i = \text{const.}$$

Equation (1), p. 2.14 opens the favorable possibility to compute the amount of a just detectable gross error in the ith observation:

$$\nabla_0\ell_i = \frac{\delta\sigma_0}{\left(q_{v_iv_i}\right)^{1/2}p_i} = \frac{\delta\sigma_0}{\left(r_ip_i\right)^{1/2}}$$

The internal reliability of a system (for bundle adjustment: image space) may be defined by these just detectable gross errors $\nabla_0\ell_i$ ($i = 1,\ldots.,n$).

With $\sigma_0^2 = p_i \sigma_{\ell_i}^2$:

$$\nabla_0 \ell_i = \sigma_{\ell_i} \frac{\delta}{(r_i)^{1/2}}$$

Criterium for good internal reliability:

$$\sup\{\nabla_0 \ell_i\} = \min$$

Since $\delta\sigma_0 = \text{const.}$:

$$\sup\left\{\frac{1}{r_i p_i}\right\} = \min$$

$$\inf\left\{\frac{1}{r_i p_i}\right\} = \max$$

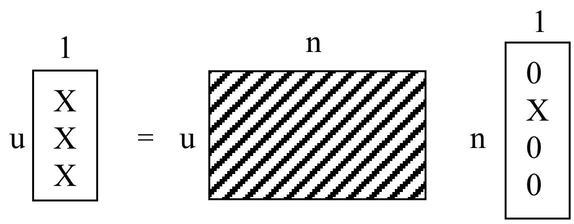
External reliability:

$$x + \nabla x = \left(A^T P A\right)^{-1} A^T P(\ell + \nabla_0 \ell) \quad , \quad \nabla_0 \ell \text{ ...non-detectable blunders}$$

$$\nabla x = \left(A^T P A\right)^{-1} A^T P \nabla_0 \ell \quad , \quad \text{if A is not disturbed by } \nabla_0 \ell$$

Assumption: One blunder in observation i $(\nabla_0 \ell_i)$

$$\nabla x_{(i)} = \left(A^T P A\right)^{-1} A^T P \nabla_0 \ell_{(i)}$$



Generally: $\nabla_0 \ell_i$ effects all components of ∇x

For i = 1,…,n we collect all corresponding vectors $\nabla x_{(i)}$ in the matrix ∇X as:

$$\nabla X = \left(\nabla x_{(1)}, \ldots, \nabla x_{(n)}\right)$$

$$\begin{array}{cc} & n \\ u & \boxed{\nabla X} \end{array}$$

For $P = \text{diag}\,(p_1, p_2, \ldots, p_n)$:

$$\nabla x_{(i)} = \left(A^T P A\right)^{-1} A^T p_i \nabla_0 \ell_{(i)}$$

$$\begin{array}{cc} & 1 \\ u & \begin{bmatrix} X \\ X \\ X \end{bmatrix} \end{array} = u \begin{array}{c} u \\ \boxed{N^{-1}} \end{array} \boxed{\;A^T\;} \begin{bmatrix} 0 \\ X \\ 0 \\ 0 \end{bmatrix}$$

See: Advantages with $P = \text{diag}$

$$= N^{-1} a_i^{(T)} p_i \nabla_0 \ell_i$$

ith column of A^T

2 Problems:

- for each $\left(\nabla_0 \ell_i\right)$ one complete vector $\nabla x_{(i)}$
  - $\succ$ very complex situation (matrix $\underset{u,n}{\nabla X}$)
- ∇X not rotation invariant

2.1.3.3 Invariance of reliability measures

Invariance of internal reliability measures:

$Q_{vv} = P^{-1} - Q_{\hat{\ell}\hat{\ell}}$, P independent of coordinate system

$Q_{\hat{\ell}\hat{\ell}} = A Q_{xx} A^T$

with $\overline{A} = A T^{-1} = A T^T$

$\overline{Q}_{xx} = T Q_{xx} T^T$

$T T^T = T^T T = I$:

$$\overline{Q}_{\hat{\ell}\hat{\ell}} = \overline{A}\,\overline{Q}_{xx}\,\overline{A}^T = A \underbrace{T^T T}_{I} Q_{xx} \underbrace{T^T T}_{I} A^T = Q_{\hat{\ell}\hat{\ell}}$$

$\succ Q_{vv}$ invariant

$v = Q_{vv} P \ell$ $\succ$ v invariant

$\succ$ Internal reliability measures invariant

$\succ$ v, Q_{VV} invariant with respect to selection of datum (bei nicht redundanten Datumsparametern)

Invariance of external reliability measures:

$$\nabla X = Q_{xx} A^T P \nabla_0 L \quad ; \quad \nabla_0 L = \left(\nabla_0 \ell_{(1)}, \ldots, \nabla_0 \ell_{(n)}\right)$$

$$\nabla \overline{X} = \overline{Q}_{xx} \overline{A}^T P \nabla_0 L = T Q_{xx} \underbrace{T^T T}_{I} A^T P \nabla_0 L$$

$$\nabla \overline{X} = T Q_{xx} A^T P \nabla_0 L$$

$$\left(\neq Q_{xx} A^T P \nabla_0 L\right) \succ \nabla X \quad \text{not rotation invariant}$$

Matrix $\nabla X^T \nabla X$ is independent of coordinate system:

$$\nabla \overline{X}^T \nabla \overline{X} = \nabla_0 L P A Q_{xx} \underbrace{T^T T}_{I} Q_{xx} A^T P \nabla_0 L = \nabla X^T \nabla X \ .$$

Square root of diagonal elements of $\nabla X^T \nabla X$:

$$\left\| \nabla x_{(i)} \right\| = \left(\nabla x_{(i)}^T \nabla x_{(i)}\right)^{1/2}$$

With P = diag:

$$\left\| \nabla x_{(i)} \right\| = p_i \nabla_0 \ell_i \left(a_i^{(T)T} N^{-1} \cdot N^{-1} a_i^{(T)}\right)^{1/2}$$

Requirement for reliable network:

$$\boxed{\sup\left\{\left\| \nabla x_{(i)} \right\|\right\} = \min \ , \quad i = 1, \ldots, n}$$

Baarda's external reliability measures:

$$\overline{\lambda}_0^{(i)} = \nabla x_{(i)}^T Q_{xx}^{-1} \nabla x_{(i)}$$

$\overline{\lambda}_0^{(i)}$ invariant with respect to translations and rotations

Compute $\overline{\lambda}_0^{(i)}$ for i = 1,…,n

Global measure for external reliability:

$$\boxed{\overline{\lambda}_0 = \sup\left\{\overline{\lambda_0}(i)\right\}}$$

2.2 Reliability structures of close-range systems

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PRECISION AND RELIABILITY ASPECTS IN CLOSE-RANGE PHOTOGRAMMETRY

Summary

The relatively small linear adjustment systems in close-range photogrammetry offer the possibility of an advanced statistical treatment. Using synthetic data, precision and reliability features of different network types are investigated. Special attention is focussed on the problem of additional parameters, introduced for systematic error compensation and on the internal and external reliability with respect to blunder detection. As a major result of these investigations a rejection procedure for non-determinable additional parameters is suggested, the extremely bad reliability structures of two photo networks are pointed out and large base four photo arrangements are recommended for professional use. In addition, external reliability measures are recommended for use as tolerance criteria in the future.

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1. Introduction

Close range photogrammetry covers a wide area of applications which sometimes differ considerably from each other with regard to instrumentation, number and arrangement of photographs, size and shape of the object, data processing, required accuracy and economic pressure for fast completion of the project. This has resulted in a great variety of processing techniques being recommended and used, often based on approximate solutions, which sacrifice both accuracy and economy. Occasionally one fails to see that when conventional photographs (based on projective relations) have to be processed analytically, a general bundle solution potentially has both flexibility and high accuracy.

In order to obtain the best possible results in high accuracy applications with a minimum of effort, optimum use must be made of the network design, the image coordinate measurements and the applied statistical model of bundle adjustment. As matters stand, the network design and the adjustment model can currently be regarded as the weakest parts in the whole problem.

Systematic investigations of precision of close-range networks (see Hell /10/) are largely lacking, compared to the situation in aerial triangulation. Similar conditions exist with respect to reliability investigations. Systematic error compensation, blunder detection and weight estimation (following the general definition of reliability, Baarda /1/) can still be considered to belong to a rudimentary status of development. This, of course, is also valid for aerial triangulation to a certain extent. So, contrary to the intentions of an Invited Paper - which usually should give an overview of the currently available and applied techniques - this paper includes mainly investigations of the author, in order to stimulate future investigations on a more extended basis.

The quality of a statistical model of the bundle adjustment is characterized by its accuracy.[1)] The concept of a c c u r a c y consists of two parts: p r e c i s i o n and r e l i a b i l i t y . Following some statistical definitions related to precision and reliability, the effects of both terms are demonstrated using different network arrangements based on synthetic data.

1) In order to conform to international terminology the term "accuracy" used here is different from that used in Grün /6/. The term "precision" in this paper corresponds to "accuracy" in /6/.

These investigations follow a line of research earlier established (Grün /6/). Special attention is focussed on the problem of additional parameters and the internal and external reliability of networks.

2. The concepts of precision and reliability in the bundle solution

Let the linear statistical model of bundle adjustment be

$$\ell - e = Ax \; ; \; P$$
$$E(\ell) = Ax \; ; \; E(e) = 0 \tag{1}$$

e...vector of true observation errors
P...weight matrix of the observation vector ℓ

with the null hypothesis H_0

$$H_0 : \ell \sim N\left(Ax, \sigma_0^2 P^{-1}\right) \tag{2}$$

(i.e., ℓ has a multidimensional normal distribution with the expectation $E(\ell) = Ax$ and the dispersion $D(\ell) = \sigma_0^2 P^{-1}$; σ_0...standard deviation of unit weight, usually to be estimated).

Suppose a minimum variance unbiased estimation of x and $\sigma_0{}^2$ is performed with

$$\hat{x} = (A^TPA)^{-1}A^TP\ell \tag{3a}$$

$$\hat{\sigma}_0{}^2 = \frac{1}{r}(A\hat{x} - \ell)^TP(A\hat{x} - \ell), \quad r = n - u \text{ (redundancy)} \tag{3b}$$

and the residuals are denoted by

$$v = A\hat{x} - \ell \tag{4}$$

Then under H_0 the distributions of $\hat{x}$ and v are

$$\hat{x} \sim N(x,K_x), \quad K_x = \sigma_0{}^2Q_{xx} \tag{5a}$$

$$v \sim N(0,K_v), \quad K_v = \sigma_0{}^2Q_{vv} \tag{5b}$$

with

$$Q_{xx} = (A^TPA)^{-1}, \quad Q_{vv} = P^{-1}-AQ_{xx}A^T. \tag{6}$$

The term "precision" describes the statistical quality of the estimated parameters $\hat{x}$, if the a-priori assumptions (functional and stochastical relations) of the adjustment model (1) are considered to be true. Hence the covariance matrix K_x contains all the information concerning the precision of the solution $\hat{x}$. The complete matrix K_x however, is usually too global a precision measure, and as such individual precision measures are necessary for individual applications.

The following very popular precision measures are obtained by using the traces of the corresponding covariance matrices:

$$\sigma_X^2 = \frac{tr(K_x^X)}{n_X} \quad , \quad \sigma_Y^2 = \frac{tr(K_x^Y)}{n_Y} \quad , \quad \sigma_Z^2 = \frac{tr(K_x^Z)}{n_Z} \qquad (7)$$

K_x^X, K_x^Y, K_x^Z...corresponding parts of K_x for X,Y,Z

n_X, n_Y, n_Z...numbers of X, Y, Z coordinates

For the relations of these measures to empirical accuracy measures, commonly used in test block investigations of aerial triangulation, see Grün /9/.

The term "reliability" defines the quality of the adjustment model with respect to the detection of model errors. Those errors can be blunders, systematic errors (mistakes in the functional assumptions) and weight errors (mistakes in the stochastical assumptions). Currently, the term reliability refers mainly to blunder detection. This is correct, since a sophisticated self-calibration concept provides for the compensation of systematic errors and since the problem of weight improvement should be treated separately (preferably by advanced methods for weight estimation). It was Baarda /1/ who developed a rather complete reliability theory which recently has also been adopted for the bundle method (Förstner /4/, Grün /6/, /7/, /8/).

The *internal reliability* gives the magnitude of a blunder in an observation ($\nabla \ell_i$) which is just non-detectable on a certain probability level. In the following equation it is assumed that only one blunder appears in the network:

$$\nabla \ell_i = \sigma_0 \frac{\delta}{p_i \sqrt{q_{v_i v_i}}} \quad , \qquad (8a)$$

δ...non-centrality parameter of the data-snooping test

p_i...weight of observation ℓ_i

$q_{v_i v_i}$...iths diagonal element of Q_{vv}

The e x t e r n a l r e l i a b i l i t y indicates the effect ot this non-detectable blunder on the estimated quantities ∇x_j .

$$\nabla x_j = (A^T PA)^{-1} A^T P \nabla \ell_i \tag{8b}$$

The effect on object space point coordinates is usually of dominant interest. So far, for bundle adjustment, the internal reliability can be considered to be defined in the image space and the external reliability in the object space.

3. Bundle solution refinement by additional parameters

The procedure of self-calibration using additional parameters (APs) is widely accepted to be the most efficient method of systematic error compensation. Polynomials have proved to be a proper device in systematic image error modeling. The functional, numerical and statistical advantages of bivariate orthogonal polynomials have been emphasized by Ebner /3/, Grün /5/. Large accuracy improvements with self-calibrating bundle adjustment have been reported in aerial triangulation (Brown /2/, Grün /5/, /9/ and others). In close-range photogrammetry the improvements are expected to be less significant since here the systematic error sources are less powerful. Yet, in order to get the best possible results out of a given problem, the self-calibration technique should become a standard procedure even in close-range applications.

In connection with the functional extension of the bundle model, however, problems arise with respect to a change of the model quality. The improper use of APs may cause serious deterioration of the results instead of expected improvements. So a check of the applied statistical model of bundle adjustment becomes necessary, usually denoted by "additional parameter testing". The significance and the determinability of APs must both be taken care of. Significance tests work within a given model, i.e. the quality of the model is accepted as it is and the formal significance of individual components or subsets is checked. Insignificant APs have to be rejected because they may only weaken the covariance matrix K_x without contributing anything positive to the functional model. Useful hints for significance testing may be found in statistical textbooks and related publications.

To the best of this author's knowledge the problem of determinability has not been treated extensively, nor has it been solved satisfactorily. Some suggestions for the simultaneous treatment of both aspects are given in Grün /5/, /6/.

In aerial triangulation systems, which usually lead to relatively large linear systems (in the order of 1,000 to 10,000 unknowns) to be solved and which require much attention in order to avoid wastage of computing time, the author has based his rejection decisions on the size of correlations created by APs. Here a correlation coefficient of 0.9 between APs and any other unknowns of the system is regarded to be already too high a value, thus leading to a rejection of the concerned APs.

In close-range systems, which normally are by far smaller, more direct approaches can be used. The most drastic one is to compare the size of standard deviations of final results (e.g. of object point coordinates) of the extended system with those of the non-extended conventional bundle solution. An extraordinary increase of those standard deviations, caused by non-determinable APs, indicate the necessity for rejection.

Assume the correct statistical model (I) to be

$$-e = Ax + Bz - \ell \; ; \quad P \tag{9a}$$

$$E(e) = 0, \quad D(e) = K_e = \sigma_0^2 P^{-1} \tag{9b}$$

z...vector of APs

This leads to

$$E(\ell) = Ax + Bz$$

$$E^I(\hat{x}) = x \tag{10a}$$

$$K^I_{x,z} = \sigma_o^2 \begin{pmatrix} A^TPA & A^TPB \\ B^TPA & B^TPB \end{pmatrix}^{-1} = \sigma_o^2 \begin{pmatrix} Q^I_{xx} & Q_{xz} \\ Q_{zx} & Q_{zz} \end{pmatrix}$$

$$K^I_x = \sigma_o^2 Q^I_{xx} \tag{10b}$$

Assume the selected (erroneous) statistical model (II) to be

$$-e = Ax - \ell \; ; \quad P \tag{11a}$$

$$E(e) = 0, \quad D(e) = K_e = \sigma_0^2 P^{-1} \tag{11b}$$

(here the APs are not modeled!)

This leads to

$E(\ell) = Ax + Bz$

(the expectation of ℓ has not changed!)

$$E^{II}(\hat{x}) = x + \left(A^TPA\right)^{-1}A^TPBz) \quad (12a)$$

$$K_x^{II} = \sigma_o{}^2(A^TPA)^{-1} = \sigma_o{}^2Q_{xx}^{II} \quad (12b)$$

Hence we get in this case (II) a biased solution $\hat{x}$ (bias: $\left(A^TPA\right)^{-1}A^TPBz$).
With $L = Q_{xx}^{II}A^TPB$ and $M = LQ_{zz}L^T$ we get the relation

$$Q_{xx}^{I} = Q_{xx}^{II} + M .$$

Since M is at least positive semidefinite, $K_x^I = \sigma_0^2Q_{xx}^I$ includes always equal or larger variances than $K_x^{II} = \sigma_0^2Q_{xx}^{II}$, due to the effect of APs in model (I). The less the difference between K_x^I and K_x^{II} the more we can be sure of not having introduced weak APs.

σ_0^2 is unbiased estimable in model (I) even if this model should be overparameterized; only if individual APs are missing, σ_0^2 is not unbiased estimable any more.

In order to demonstrate the effect of APs on the precision of object point coordinates the practical example already used in Grün /6/ is introduced here again (synthetical data).

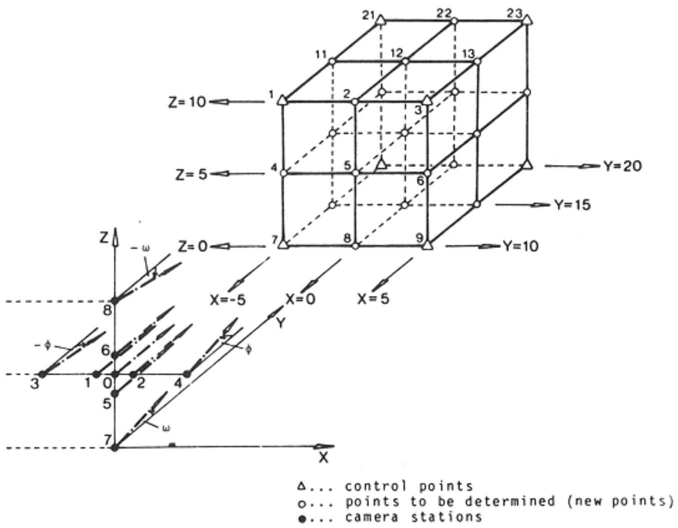

Figure 1. Synthetic network arrangement for the demonstration of precision and reliability features.

A cube has been assumed containing 27 regularly distributed points; 8 of them serve as control points, the rest are new points. The camera stations are denoted by nos. 0,...,8. The photos 0,1,2,5,6 are "truly vertical" (small base: 2 m); the photos 3, 4 are convergent ($\phi \approx 20.5^g$), nos 7, 8 are tilted ($\omega \approx 20.5^g$), both sets include a large base (10 m).

Five different network arrangements are investigated:

Version	Photos	
A	1,2	1 small base, 2 vertical photos
B	3,4	1 large base, 2 convergent photos
C	3,0,4	3 photos, common base direction
D	1,2,5,6	2 small bases, perpendicular to each other
E	3,4,7,8	2 large bases, perpendicular to each other

The object space coordinates of the 8 control points are assumed to be free of errors, the image coordinates to be uncorrelated and of equal precision (equal weights). APs are introduced as free unknowns. The AP set used here is the bivariate orthogonal one introduced by Ebner /3/. Since a wide object range in Y-direction (see Figure 1: ΔY_{max} = 10 m) compared with the average exposure distance (Y_0 = 15 m) is used, this AP set is extended by 3 parameters of the interior orientation. Those APs might be useful in many close-range applications and as such interesting to introduce and to investigate.
The 15 APs which we get finally are

(with $k = x^2 - \frac{2b^2}{3}$, $l = y^2 - \frac{2b^2}{3}$):

$$\Delta x = b_1 + 0 + b_3 \frac{x}{c} + b_4 x + b_5 y - b_6 2k + b_7 xy + b_8 l + \ldots$$

$$\Delta y = 0 + b_2 + b_3 \frac{y}{c} - b_4 y + b_5 x + b_6 xy - b_7 2l + 0 + \ldots$$

$$(\Delta x)\ldots + 0 + b_{10} xl + 0 + b_{12} yk + 0 + b_{14} kl + 0 ;$$

$$(\Delta y)\ldots + b_9 k + 0 + b_{11} yk + 0 + b_{13} xl + 0 + b_{15} kl ; \qquad (13)$$

Table 1 shows the results of the computations. The mean standard deviations of object space coordinates and exterior orientation elements are related to σ_0 = 1 µm and are given in (mm) and (c) respectively. In order to be able to understand the effects of APs the maximum correlation coefficients are indicated in each computation version (correlations between object point coordinates, NP/NP, between object point coordinates and APs, NP/AP, between exterior orientation elements and APs, EO/AP, between APs themselves, AP/AP). In addition, the APs which cause such maximum correlations are indicated.

Analyzing the results according to their sequence in Table 1 leads first to the 2-photo arrangement 1/2 (A, small base). Here the use of all 15 APs yields very bad results in X and Z (correlations $\sim$ 1.00), and better though not good enough results in "depth" Y. After the rejection of b_7 (highest AP correlation) the Z-coordinates improve significantly. The same appears with respect to the X-coordinates after rejecting b_1. A further improvement in X by a factor 1.3 is obtained by rejecting b_{14}. The remaining 12 APs lead to a homogeneous precision in X,Z, which is not much worse than the best possible precision, indicated by the 0-version (which does not include the influence of APs). It is interesting to note that the camera constant (b_3) leads to high correlations (0.98) with the Y-coordinates of the perspective centers, although it does not really deteriorate the object point values. Startling as well, are the high correlations (0,98) between coordinates of the object points even in the 0-version, which demonstrates the unfavorable precision structure of this network type.

Network 3/4 (B, large base) starts with much better results in the 15 APs version. The rejection of b_8 and b_7, however, give a remarkable further improvement in X and Z respectively. As in network A the depth Y is relatively stable from the very beginning, but has improved by a factor of 4.8. The better precision structure of network B shows up also in the 0-version by an improvement of a factor 2.5 against the network A, which is half the value of the base ratio ($b_{3/4}$: $b_{1/2}$ = 10:2). Again the camera constant (b_3) influences mainly the perspective center coordinates Y_0. This effect is explicitely shown in network 3/0/4 (C), where b_3 is excluded in the fourth computational version. This procedure does not affect the coordinates of the object points much, but reduces the mean standard deviation of Y_0 by a factor of 3.0. It would be worthwhile to mention an interesting fact here. Due to the special photo arrangements, the Y-coordinates of the object points in all networks have the worst precision, but just the opposite is valid for the Y_0-coordinates of the perspective centers, which are best determinable, compared with X_0,Z_0. In the 4-photo version 1/2/5/6 (D, small bases) and 3/4/7/8 (E, large bases) the interior orientation elements b_1,b_2,b_3 are excluded, which, however, does not lead to much improvement (factors 1.2 for X and Z). Only the version E yields optimum precision results, with and without the complete set of 15 APs.

In order to obtain a better overview, the mean standard deviations of the object point coordinates of all versions are graphed against the number of APs in the Figure 2. Although all large base versions show relatively good precision features in their 0-versions, only version E gives optimum results with the full AP set. This version is definitely to be preferred.

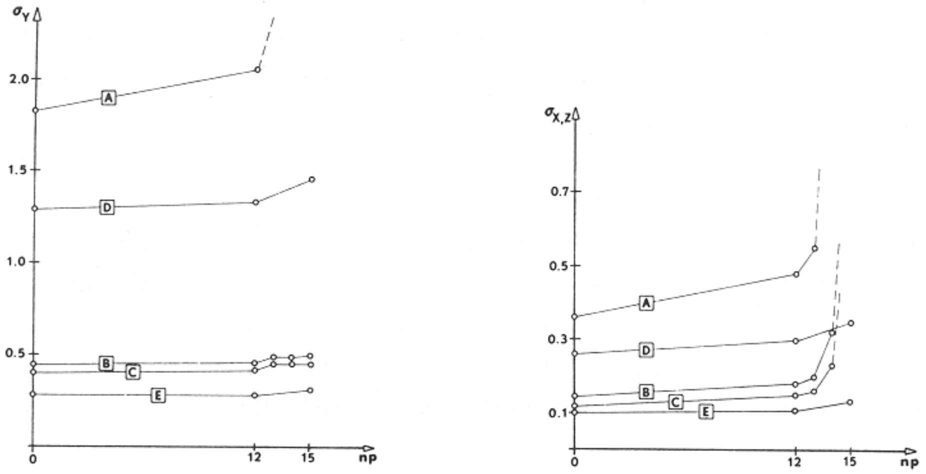

Figure 2. Precision measures of networks A, B, C, D, E in different additional parameter versions (np = number of APs).

These investigations show very clearly the necessity for an individual and sophisticated statistical treatment of the APs in a rejection procedure. It is trivial to state that in different networks different APs might have to be rejected. The rejection strategy applied here in networks A,B,C was based on correlation checks. Those parameters which cause the largest correlations with respect to the object point coordinates and to the other APs have been rejected. This procedure was found to work quite well and it can really be recommended for professional use.

What should be considered to be a "high" correlation is still in question. In this connection it is interesting to see that the rejection of an AP which was correlating at the 90 %-level (b_{14} in A and B) leads to a precision improvement of a factor 1.3. Experience in this matter indicates that in close-range photogrammetry an individual AP component cannot be expected to improve the results by a factor of 1.3. So the author recommends that an AP which has 0.9 or even higher correlations with other decisive components of the system should be excluded.

The relatively small systems in close-range photogrammetry permit computations, which might not be feasible in aerial triangulation. So apart from correlation control the check of a change of the standard deviations of all individual object points also becomes possible. Consequently, a complete and successful system control is relatively easy to perform, thus leading to highly precise and more reliable results.

4. The internal and external reliability of bundle systems

If correctly applied, self-calibration provides for the compensation of the systematic errors. Weight estimation is a separate problem, not touched in this paper. Consequently, for reliability studies, only the reliability of our systems with respect to blunder detection has to be investigated. In Grün /6/ internal reliability investigations have been carried out using the above mentioned synthetic example. It was clearly emphasized that in order to obtain reliable systems a four photo coverage should be used with two bases perpendicular to each other to avoid pure epipolar plane observations.

Here, these internal reliability studies are picked up again but presented in greater detail (separation into control and non-control observations, indication of the just non-detectable blunders). In addition, external reliability studies are added, referring to the same networks and data sets. The author is grateful to cand. ing. W. Przibylla, who provided many of the results presented in this chapter through his diploma thesis /11/. Based on the experiences reported in Grün /8/ (good reliability features of "twin ray" observations) the image coordinates of network 3/0/4 are here considered to be measured in stereomode (photos 3/0 and 0/4), so that all image points of photo 0 are observed twice, thus leading to "twin rays". So the complete set of rays of network 3/0/4 (CS...version C, measured in stereomode) is separated into "inner rays" (twin) and "outer rays" (single).

Table 2 shows the internal reliability results.

Global internal reliability indicators are defined as

$$Ri(x) = \frac{tr(Q^x_{vv})}{n} \quad , \qquad Ri(y) = \frac{tr(Q^y_{vv})}{n} \tag{14a}$$

$$\nabla l^x_M = \frac{1}{n} \Sigma \, \nabla l^x_i \qquad \nabla l^y_M = \frac{1}{n} \Sigma \, \nabla l^y_i \tag{14b}$$

$tr(Q^x_{vv})$, $tr(Q^y_{vv})$...subtraces of Q_{vv}, related to x- and y-residuals.

∇l^x_M, ∇l^y_M...average size of non-detectable blunders in x- and y-observations

n...corresponding number of observations.

The values ∇l^x_M, ∇l^y_M are related to σ_o = 1µm, δ_i = 4.1, α_o = 0.001, β_o = 0.80. The internal reliability measures are subdivided into control and non-control values, as both types show significant differences between them. Additionally, in network 3/0/4 (version CS) the inner and outer ray observations are differentiated.

Table 2: Global internal reliability measures.

Version	r	Control Ri(x)	Control Ri(y)	Control ∇l^x_M (µm)	Control ∇l^y_M (µm)	Non-control Ri(x)	Non-control Ri(y)	Non-control ∇l^x_M (µm)	Non-control ∇l^y_M (µm)
A	69	0.66	0.71	5.2	5.0	0.00	0.45	→ ∞	6.2
B	69	0.66	0.71	5.2	5.0	0.00	0.44	→ ∞	6.2
CS i.r.	141	0.86	0.86	4.5	4.5	0.70	0.71	4.9	4.9
CS o.r.		0.71	0.77	4.9	4.7	0.25	0.68	8.4	5.0
D	135	0.76	0.76	4.8	4.8	0.56	0.56	5.6	5.6
E	135	0.76	0.76	4.8	4.8	0.56	0.56	5.6	5.6

r...redundancy
i.r...inner rays
o.r...outer rays

Control observations are reliable in all versions, there is only a minor difference between x- and y- observations. The average values for just non-detectable blunders are fairly homogeneous, ranging from 4.5 σ_o to 5.2 σ_o.

The reliability values of non-control observations of the individual networks, however, differ significantly from each other. The x-observations of versions A and B are not checked at all, the y-observations can be considered to be of average reliability.

Things change considerably when a third photo is introduced in a suitable location. Although the x - values of the outer rays of versions CS are still weak ($\nabla l_M^x = 8.4\sigma_0$), fairly good results are obtained for the inner ray x-observations ($\nabla l_M^x = 4.9\sigma_0$).

Sufficiently good and homogeneous results in x and y are only achieved with the versions D and E, i.e. with the 4 photo arrangements ($\nabla l_M^x = \nabla l_M^y = 5.6\sigma_0$).

The external reliability measures show the effect of a just non-detectable blunder onto the solution vector. As indicated by equation (8b) the blunder $\nabla \ell_i$ is propagated to ∇x_j by the function $(A^TPA)^{-1}A^TP$. So the network design, and in this context specially the normal inverse $(A^TPA)^{-1}$, has an essential influence on the quality of the external reliability. Since this inverse (together with $\sigma_0{}^2$) defines also the precision of a system, a direct relation between external reliability and precision becomes obvious. For this reason quite different values of external reliability can be expected in different networks, even if the internal reliability values are identical (compare Tables 2,3, versions D, E). Consequently the internal reliability measures may, in general, not be sufficient to describe the reliability of a network. For this purpose, only the measures for the external reliability can serve as indicators.

To support these statements the effects of all individual just non-detectable blunders of the networks A, B, CS, D, E (see Table 2) on the object point coordinates have been computed. Table 3 shows the results. For simplification only the maximum values for ∇XZ, ∇Y[1] are indicated. As the effects of just non-detectable blunders have a reasonable statistical basis (derived from hypotheses testing - datasnooping) and since they represent error limits (based on probability levels α_0, β_0) they can advantageously be used as tolerance values. Although standard deviations are preferred for use as precision measures among surveyors, they have never been accepted by engineers in connected disciplines and much confusion in communication has resulted therefrom.

Another restrictive feature of standard deviations, which makes them less suitable for use in practical projects, results from the fact that they are based on the assumption of random errors only, which is often an unrealistic assumption, as practice shows. Now these problems seem likely to be overcome by the use of external reliability measures as tolerances. Consequently the values of Table 3 are denoted by "tolerances", and three types are distinguished:

1) In the following the object point coordinates are denoted by large letters X, Z, Y.

- 1st order tolerances: Values for non-control points, caused by blunders in observations belonging to these points.
- 2nd order tolerances: Values for non-control points, caused by blunders in control observations.
- 3rd order tolerances: Values for non-control points, caused by blunders in observations of other non-control points.

All measures ∇XZ, ∇Y have been obtained by assuming one blunder only at a time.

Table 3. External reliability measures for object point coordinates (maximum tolerances based on $\alpha_o = 0.001$, $\beta_o = 0.80$, $\sigma_o = 1$ µm).

Version	1st order tolerances ∇XZ_{Max} (mm)	∇Y_{Max} (mm)	2nd order tolerances ∇XZ_{Max} (mm)	∇Y_{Max} (mm)	3rd order tolerances ∇XZ_{Max} (mm)	∇Y_{Max} (mm)
A(1/2)	$\rightarrow \infty$	$\rightarrow \infty$	0.58	1.81	0.17	0.69
B(3/4)	$\rightarrow \infty$	$\rightarrow \infty$	0.24	0.40	0.05	0.13
CS(3/0/4)	1.45	4.07	0.18	0.37	0.04	0.11
D(1/2/5/6)	1.81	6.13	0.25	0.80	0.09	0.33
E(3/4/7/8)	0.64	1.30	0.12	0.17	0.02	0.07

∇XZ_{Max}...Maximum value in X or Z, if both coordinates X,Z are considered together.

As could be expected, the maximum distortion of object space coordinates is always caused by a blunder in the observations of the corresponding point (1st order tolerances). Compared to these values the effects of just non-detectable blunders in image coordinate observations of control points (2nd order tolerances) are relatively small, although large enough not to be ignored, especially in the small base versions A and D.

The maximum values for the 3rd order tolerances are comparably small in all networks, however, it can happen that for individual points the 3rd order tolerances exceed the 1st order tolerances. This means, that the effects of non-detectable blunders in observations other than those of the point, may exceed the effects of blunders in observations of the point itself.

Comparing A with D and B with E one sees that a doubling of the number of photos without changing the size of the bases leads to an improvement of the maximum 2nd and 3rd order tolerances by a factor of 2.

Keeping the number of photos constant, a base enlargement does not improve the internal reliability. The external reliability, however, is improved significantly (average improvement factors of 3 and 5 for XZ and Y between the networks A and B, D and E). Additionally, the values for the X, Z and Y coordinates become more homogeneous. Generally it can be stated that with respect to reliability features (here related to blunder detection), the networks A and B (2 photo versions) are completely failing, as the maximum 1st order tolerances result in values close to infinity.

5. Concluding remarks

In order to obtain highly precise and reliable photogrammetric networks a few basic requirements have to be met. These are easy to fulfill in practical projects.

For a long time it has been well known that the use of large bases (for complete object coverage mostly connected with convergent photography) leads to a much better object point coordinate precision than the small base concept. However, precision measures are not of much value, if the reliability of a network is bad.

In order to get practical network results closer to theoretical precision measures, primary efforts must be directed towards the compensation of systematic image errors, which can preferrably be achieved by self-calibration. The extension of the bundle model by additional parameters, however, more or less changes the network's precision structure. Further, to avoid a serious deterioration of the precision of required components (e.g. object point coordinates) those additional parameters which weaken the system must be excluded. In close-range systems this can be achieved by correlation control and variance check, as presented in this paper. Equally important is a design of networks, which provides for good blunder detection properties. To obtain sufficient redundancy for blunder control at least a fourfold photo coverage should be aspired for. Ray arrangements leading to one common epipolar plane only have to be avoided.
Although the internal reliability measures are useful indicators for a network's reliability properties (especially in extreme cases), a comprehensive reliability control can only be performed by analyzing the external reliability. External reliability measures have a close relation to precision measures. This permits the general one-way statement to be made: If a network is designed with excellent external reliability features, its precision can also be considered to be

outstanding. Network E of our examples has such properties.

In addition, external reliability measures can be regarded as tolerance values, which enable easier communication between photogrammetrists and the users of their products, than was possible in the past.

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TABLE 1: Precision measures for different networks (related to $\sigma_0 = 1$ μm) 7

Photo arrangement	Add. parameter version	Mean standard dev. of non-control points σX (mm)	σZ (mm)	σY (mm)	Mean standard deviations of exterior orientation elements σX_0 (mm)	σZ_0 (mm)	σY_0 (mm)	$\sigma \phi$ (c)	$\sigma \omega$ (c)	$\sigma \kappa$ (c)	Maximal corr. coefficients NP/NP NP/AP	EO/AP AP/AP
1/2 A	1-15	91.2	95.1	2.4	254	266	0.76	0.20	150	0.10	~1.00 ~1.00	~1.00 2/7~1.00
	1-6 8-15	90.5	0.52	2.4	253	0.21	0.45	0.18	0.09	0.08	~1.00 ~1.00	X_o/1~1.00 1/5/8=-0.90
	2-6 8-15	0.61	0.50	2.4	0.36	0.21	0.45	0.18	0.09	0.05	0.98 14:0.90	Y_o/3=0.98 5/14=0.79
	2-6 8-13 15	0.48	0.48	2.1	0.31	0.21	0.45	0.13	0.09	0.05	0.98 8:0.78	Y_o/3=0.98 3/8=0.77
	0	0.36	0.36	1.8	0.16	0.13	0.06	0.07	0.06	0.04	0.98 —	— —
3/4 B	1-15	2.1	0.39	0.50	5.5	1.0	1.8	0.36	0.59	0.18	~1.00 8:0.99	X_o/8=0.998 1/8=0.995
	1-7 9-15	0.22	0.39	0.49	0.36	1.0	0.30	0.14	0.59	0.18	~1.00 7:-0.98	ω/7=0.98 1/14=-0.90
	1-6 9-15	0.22	0.18	0.49	0.36	0.19	0.30	0.14	0.12	0.05	0.88 14:-0.92	Y_o/3=0.93 1/14=-0.90
	1-6 9-13 15	0.17	0.18	0.45	0.24	0.19	0.30	0.12	0.12	0.05	0.70 10:0.80	Y_o/3=0.93 9/15=0.75
	0	0.14	0.15	0.44	0.14	0.15	0.09	0.06	0.06	0.04	0.70 —	— —
3/0/4 C	1-15	1.3	0.29	0.45	3.6	0.71	0.86	0.23	0.38	0.09	~1.00 8:0.99	X_o/8=0.999 1/8=0.998
	1-7 9-15	0.16	0.29	0.45	0.21	0.71	0.27	0.10	0.38	0.09	0.96 7:-0.97	Z_o/7=-0.97 2/7=0.75
	1-6 9-15	0.16	0.16	0.45	0.21	0.16	0.27	0.10	0.09	0.04	0.68 14:-0.72	Y_o/3=0.97 3/10=0.69
	1-2 4-6 9-15	0.15	0.15	0.42	0.18	0.16	0.09	0.10	0.09	0.04	-0.73 14:-0.72	ω/2=0.69 1/14=-0.61
	0	0.12	0.12	0.40	0.13	0.13	0.08	0.06	0.06	0.04	0.77 —	— —
1/2/5/6 D	1-15	0.35	0.35	1.5	0.26	0.26	0.26	0.11	0.11	0.04	0.98 6.7:-0.72	Y_o/3=0.96 1/8=2/9=0.75
	4-15	0.30	0.30	1.3	0.24	0.24	0.07	0.10	0.10	0.04	0.98 6:-0.79	Z_o/4=-0.78 <0.5
	0	0.26	0.26	1.3	0.12	0.12	0.06	0.06	0.06	0.03	0.98 —	— —
3/4/7/8 E	1-15	0.13	0.13	0.31	0.19	0.19	0.19	0.11	0.11	0.05	0.60 7:-0.66	Y_o/3=0.88 8/14=9/15=0.7
	4-15	0.11	0.11	0.28	0.14	0.14	0.09	0.06	0.06	0.04	0.62 <0.5	ω/7=0.62 9/15=0.75
	0	0.10	0.10	0.28	0.12	0.12	0.08	0.05	0.05	0.04	<0.5 —	— —